11447 | AP CALCULUS | SOLUTIONS

#1 (a) $x(t) = \int v(t)\,dt = \int 3t^2 - 2t - 1\,dt = t^3 - t^2 - t + c$

Given $x(2) = 5 \Rightarrow 5 = 8 - 4 - 2 + c \Rightarrow c = 3$

$\therefore \boxed{x(t) = t^3 - t^2 - t + 3}$ — Answer.

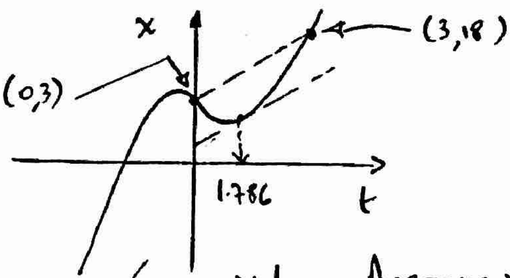
(b) Using the mean value theorem:

average velocity $= \left(\dfrac{d\,x(t)}{dt}\right)_{av} = \dfrac{x(3) - x(0)}{3 - 0} = \dfrac{18 - 3}{3} = 5$

when $3t^2 - 2t - 1 = 5$

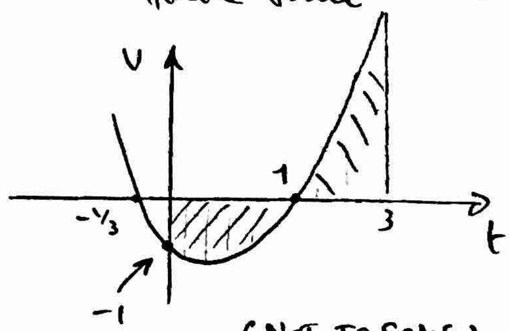
$3t^2 - 2t - 6 = 0$

In $[0,3] \Rightarrow \boxed{t = 1.786}$ (3dp)



Note: Average velocity is $\dfrac{\text{total displacement}}{\text{total time}}$

(c) Total distance is the area under the v-t graph.



(NOT TO SCALE)

Total distance $= \int_0^1 -3t^2 + 2t + 1\,dt + \int_1^3 3t^2 - 2t - 1\,dt$

$= \left(-t^3 + t^2 + t\right)_0^1 + \left(t^3 - t^2 - t\right)_1^3$

$= 1 + 15 + 1$

$= \boxed{17}$

-OR-

Particle moves to the left for first second (or time unit)

Total distance $= x(3) - x(1) + |x(0) - x(1)|$

$= 18 - 2 + |3 - 2|$

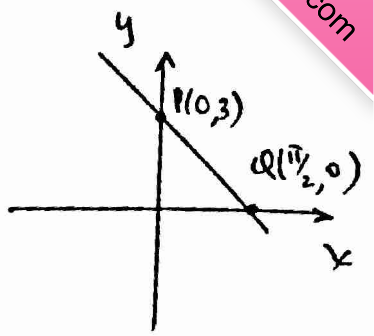
$= \boxed{17}$

#2 (a) $m = \dfrac{3-0}{0-\pi/2} = -6/\pi$ (slope of PQ)

$\Rightarrow y - 3 = -\dfrac{6}{\pi}(x-0)$ using $y - y_1 = m(x - x_1)$

$\pi y - 3\pi + 6x = 0$

$\boxed{6x + \pi y - 3\pi = 0}$ —— Answer



(b) At Q. Slope of line tangent $= f'(\pi/2)$

$\Rightarrow f'(x) = -3\sin x$

$f'(\pi/2) = -3$

Tangent line: $y - 0 = -3(x - \pi/2)$

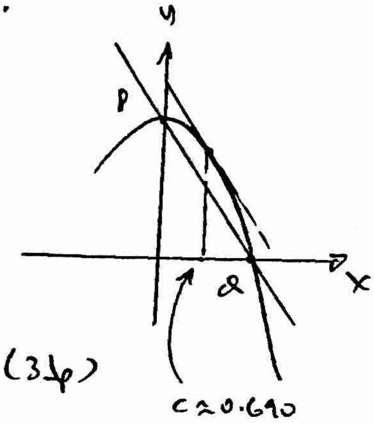
$\Rightarrow \boxed{3x + y - 3\pi/2 = 0}$ —— Answer.

(c) Using the Mean-Value Theorem:

$f'(x)_{\text{average}} = -3\sin c = -6/\pi$

$\sin c = 2/\pi$

$c = \arcsin(2/\pi) \approx \boxed{0.690}$ (3dp)

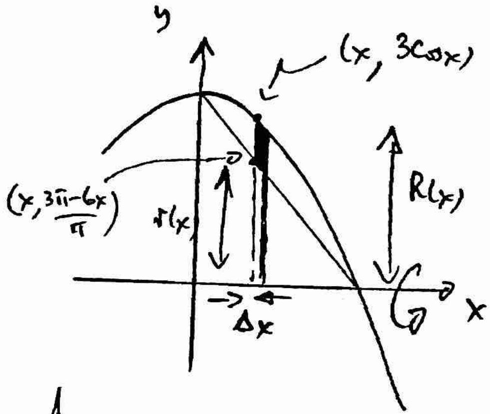


(d)

$V = \pi \int_a^b y^2 \, dx$ (Disc)

$V = \pi \int_a^b [R(x)]^2 - [r(x)]^2 \, dx$ (washer)

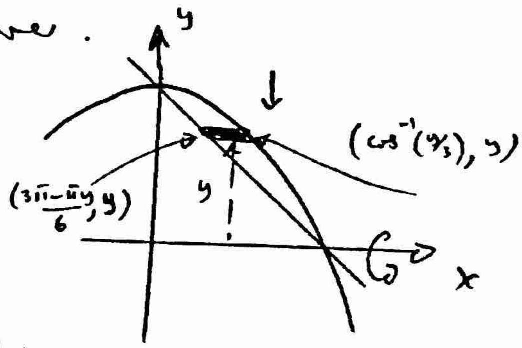
$$\Rightarrow \boxed{V = \pi \int_0^{\pi/2} 9\cos^2 x - \frac{(3\pi - 6x)^2}{\pi^2} \, dx}$$ —— Answer.



– OR –

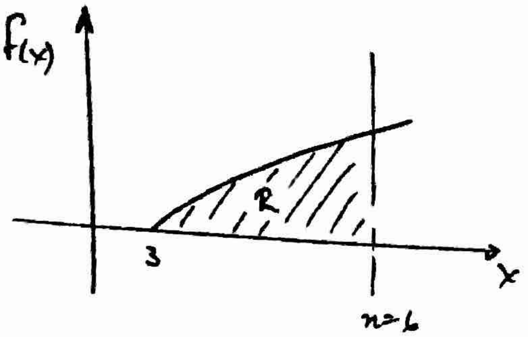
By THE SHELL METHOD:

$$V = 2\pi \int_0^3 y\left(\cos^{-1}(y/3) - \left(\frac{3\pi - \pi y}{6}\right)\right) dy$$ —— Answer.



#3

(a)



(b) $A = \int_3^6 \sqrt{x-3}\,dx = \left[\frac{2}{3}(x-3)^{3/2}\right]_3^6$

$A = \frac{2}{3}(3^{3/2}) - 0 = \boxed{2\sqrt{3}}$ — Answer:

(c) $A(w) = \int_3^w (x-3)^{1/2}\,dx$

(d) $\frac{dA(w)}{dw} = \frac{d}{dw}\int_3^w (x-3)^{1/2}\,dx = (w-3)^{1/2}$ $\left\{ 2^{nd}\text{ Fund. Theorem of Calculus} \right\}$

when $w=6$, $dA(w)/dw = \sqrt{3}$

#4 (a) $f(x) = x^3 - 6x^2 + p$

$\boxed{f'(x) = 3x^2 - 12x}$

$f'(x) = 0$, $3x(x-4) = 0$

$x = 0 \cup x = 4$

1ST DERIVATIVE TEST

$f'(x)$: + (0) − (4) +

0: Rel Max; 4: Rel Min

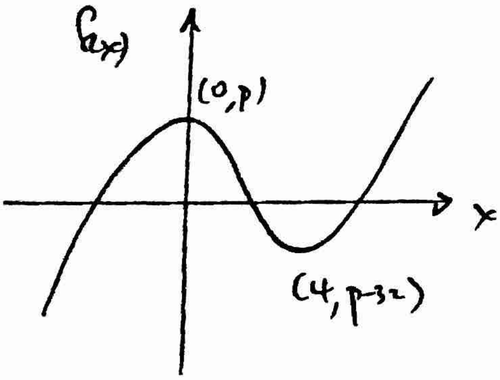
when $x=0$, $f(0) = p$

when $x=4$, $f(4) = 64 - 96 + p = p - 32$

$\Rightarrow$ Rel Min: $(4, p-32)$; Rel Max: $(0, p)$ — Answer:

i.e $\boxed{f(x)_{min} = p - 32}$; $\boxed{f(x)_{max} = p}$

(b)



$f(x)$ has 3 distinct real roots

if $p > 0 \cap p - 32 < 0$

i.e $p > 0 \wedge p < 32$

or $\boxed{0 < p < 32}$

#4 (CONT...)

(c) Using $f(x)_{av} = \dfrac{\int_a^b f(x)\,dx}{b-a}$

$\Rightarrow \quad 1 = \dfrac{\int_{-1}^{2} x^3 - 6x^2 + p\,dx}{2-(-1)}$

$3 = \left(x^4/4 - 2x^3 + px \right)_{-1}^{2}$

$3 = (4 - 16 + 2p) - (1/4 + 2 - p)$

$\Rightarrow \quad 3p = 69/4$

$\Rightarrow \quad \boxed{p = 23/4}$ — Answer.

#5 (a) $g(x) = \int_0^x f(t)\,dt \Rightarrow g'(x) = \dfrac{d}{dx}\int_0^x f(t)\,dt = f(x)$

(2nd Fund. Theorem)

$\Rightarrow \quad g(3) = \int_0^3 f(t)\,dt$

$= \dfrac{\pi(2)^2}{4} - \dfrac{1}{2}$ (Geometry)

$= \boxed{\pi - 1/2}$ — Answer.

(b) Using the first derivative test and $g'(x) = f(x)$

$g'(x)$: -2 (+) 2 (−) 4 (+) 5

Rel. Max.

$\Rightarrow$ Rel. Max. @ $\boxed{x = 2}$ — Answer.

(c) At $x = 3$, $g(3) = \pi - 1/2$ & $g'(3) = -1$

Using $y - y_1 = m(x - x_1)$

$\Rightarrow \quad y - (\pi - 1/2) = -1(x - 3)$

gives $\boxed{x + y - \pi - 5/2 = 0}$ — Answer.

(d) Using the slope of $f(x)$ or $g'(x)$:

$\Rightarrow$ $g''(x)$:

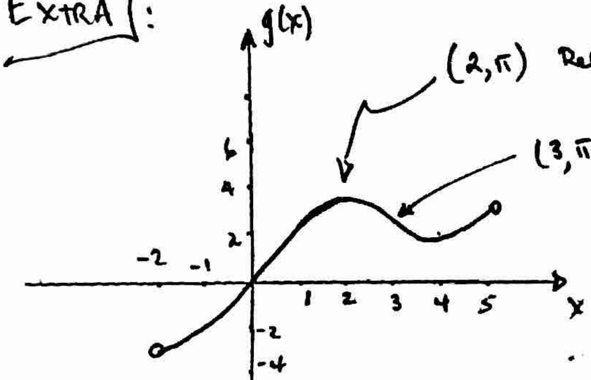
-2		0		2		3		5
o	$+$	●	$-$	$\not{o}$	$-$	$\not{o}$	$+$	o
	c. up		c. Down		c. Down		c. up	

$g''(0) = 0$ $\qquad$ $g''(3)$ is undefined.

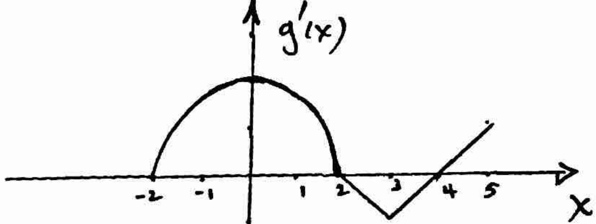
Since $g(0)$ & $g(3)$ Exist

$\Rightarrow$ Points of inflexion occur @ $\boxed{x = 0 \ , \ x = 3}$

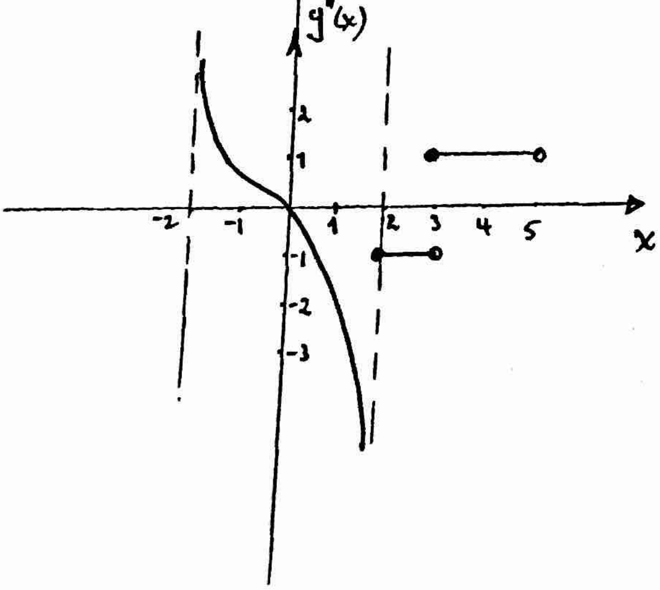
EXTRA:



$$g(x) = \begin{cases} 2\arcsin(x/2) + \dfrac{x\sqrt{4-x^2}}{2} & -2 < x < 2 \\ \pi - 2 + 2x - x^2/2 & 2 \le x < 3 \\ \pi + 7 + x^2/2 - 4x & 3 \le x < 5 \end{cases}$$



$$f(x) = g'(x) = \begin{cases} (4-x^2)^{1/2} & -2 < x < 2 \\ 2 - x & 2 \le x < 3 \\ x - 4 & 3 \le x < 5 \end{cases}$$



$$g''(x) = \begin{cases} -x(4-x^2)^{-1/2} & -2 < x < 2 \\ -1 & 2 \le x < 3 \\ 1 & 3 \le x < 5 \end{cases}$$

Note: $G(x)$ is calculated on the basis: $G(x) = \int_0^x f(t)\,dt$.

#6 (a) $\frac{dv}{dt} = -2v - 32$ $\qquad v(0) = -50$

$$\int \frac{dv}{-2v-32} = \int dt$$

* Note: Downwards velocity is taken as negative.

$$-\frac{1}{2}\ln|-2v-32| = t + c$$

Using $t=0$, $v=-50$ $\Rightarrow$ $-\frac{1}{2}\ln|68| = c$

$$\therefore \quad \frac{1}{2}\ln|68| - \frac{1}{2}\ln|-2v-32| = t$$

$$\ln\left|\frac{-68}{2v+32}\right| = 2t$$

$$\frac{-68}{2v+32} = e^{2t}$$

$$\Rightarrow \quad 2v + 32 = -68e^{-2t}$$

$$\boxed{v = -16 - 34e^{-2t}}$$ Answer.

(b) $V_{TERM} = \lim_{t \to \infty} v(t) = \boxed{-16 \text{ ft/sec}}$

(c) when $v = 20$,

$$\Rightarrow \quad -20 = -16 - 34e^{-2t} \quad *$$

$$34e^{-2t} = 4$$

$$e^{-2t} = 2/17$$

$$-2t\ln e = \ln(2/17)$$

$$t = \frac{\ln(2/17)}{-2} \approx \boxed{1.070 \text{ seconds}}$$ Answer